# The category of reflexive modules over one-dimensional Cohen-Macaulay rings

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# 1. Introduction

Let

- $R$ a commutative Noetherian ring with $(S_2)$ and $\mathrm{Q}(R)$ is Gorenstein
- $\mathrm{Mod}\, R$ the category of $R$-modules
- $\mathrm{mod}\, R$ the subcategory of $\mathrm{Mod}\, R$ consisting of finitely generated $R$-modules

For $M \in \mathrm{mod}\, R$,

$M$ is a reflexive $R$-module $\overset{def}{\Longleftrightarrow}$ the natural map $M \to M^{**}$ is an isomorphism

$\Longleftrightarrow$ $M_\mathfrak{p}$ is reflexive for $\mathfrak{p} \in \mathrm{Spec}\, R$ s.t. $\dim R_\mathfrak{p} = 1$ and $M$ satisfies $(S_2)$

where $(-)^* = \mathrm{Hom}_R(-, R)$ and

$$M \text{ satisfies } (S_2) \overset{def}{\Longleftrightarrow} \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \inf\{2, \dim R_\mathfrak{p}\} \text{ for } \forall \mathfrak{p} \in \mathrm{Spec}\, R.$$

In what follows, let

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$, $\mathrm{Q}(R)$ is Gorenstein, and $|R/\mathfrak{m}| = \infty$
- $R \subseteq A \subseteq \mathrm{Q}(R)$ an intermediate ring s.t. $A \in \operatorname{mod} R$
- $\mathrm{CM}(A)$ the subcategory of $\operatorname{mod} A$ consisting of MCM $A$-modules
- $\mathrm{Ref}(A)$ the subcategory of $\operatorname{mod} A$ consisting of reflexive $A$-modules

For $M \in \operatorname{mod} A$,

$M$ is a MCM $A$-module $\overset{def}{\iff}$ $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geq \dim A_\mathfrak{p}$ for $\forall \mathfrak{p} \in \operatorname{Spec} A$

$\iff$ $M$ is a torsion-free $A$-module.

Then $\mathrm{Ref}(A) \subseteq \mathrm{CM}(A)$ and

$$
\begin{aligned}
\mathrm{Ref}(A) &= \{M \in \operatorname{mod} A \mid \exists\ 0 \to M \to F_0 \to F_1 \text{ s.t. } F_i \in \operatorname{mod} A \text{ is free}\} \\
&= \{M \in \operatorname{mod} A \mid \exists\ 0 \to M \to F \to X \to 0 \text{ s.t. } F \text{ is free, } X \in \mathrm{CM}(A)\} \\
&= \Omega\mathrm{CM}(A).
\end{aligned}
$$

Note that $\Omega\mathrm{CM}(A) = \mathrm{CM}(A) \iff A$ is a Gorenstein ring.

By setting $E = \mathrm{End}_R(\mathfrak{m}) \cong \mathfrak{m} : \mathfrak{m}$, we have

**Theorem 1.1 (Goto-Matsuoka-Phuong)**

$$\Omega\mathrm{CM}(E) = \mathrm{CM}(E) \iff R \textit{ is almost Gorenstein and } \mathfrak{m} \textit{ is stable.}$$

Recall that

- an ideal $I$ of $R$ is stable if $I^2 = aI$ for $\exists\, a \in I$
- $\mathfrak{m}$ is stable $\iff$ $R$ has minimal multiplicity, i.e., $\mathrm{e}(R) = v(R)$
- $R$ is an almost Gorenstein ring if $\mathfrak{m}K \subseteq R$, where $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$.

Let $\Omega\mathrm{CM}_{\mathcal{P}}(R) = \{M \in \Omega\mathrm{CM}(R) \mid M \text{ doesn't have free summands}\}$.

**Theorem 1.2 (Kobayashi)**

(1) $\Omega\mathrm{CM}(E) \subseteq \Omega\mathrm{CM}_{\mathcal{P}}(R) \subseteq \mathrm{CM}(E)$.

(2) $\Omega\mathrm{CM}(E) = \Omega\mathrm{CM}_{\mathcal{P}}(R) \iff \mathfrak{m}$ *is stable.*

(3) $\Omega\mathrm{CM}_{\mathcal{P}}(R) = \mathrm{CM}(E) \iff R$ *is an almost Gorenstein ring.*

**Question 1.3**

What happens if we take $\mathrm{End}_R(I)$?

Another motivation comes from the following.

**Theorem 1.4 (Dao-Iyama-Takahashi-Vial)**

*Let $(A, \mathfrak{m})$ be an excellent henselian local normal domain with $\dim A = 2$ and $A/\mathfrak{m}$ is algebraically closed. Then*

$$A \text{ has a rational singularity} \iff \Omega\mathrm{CM}(A) \text{ is of finite type.}$$

A subcategory $\mathcal{X}$ of $\mathrm{mod}\, A$ is called of finite type if $\mathcal{X} = \mathrm{add}_A M$ for $\exists M \in \mathrm{mod}\, A$.

## Question 1.5

When is $\Omega\mathrm{CM}(R)$ of finite type for a one-dimensional ring $R$?

Recall that

- $R$ is an almost Gorenstein ring $\iff \Omega\mathrm{CM}_{\mathcal{P}}(R) = \mathrm{CM}(E)$
- $\Omega\mathrm{CM}_{\mathcal{P}}(R) = \{M \in \Omega\mathrm{CM}(R) \mid M$ doesn't have free summands$\}$.

## Corollary 1.6 (Kobayashi)

*Suppose that $R$ is an almost Gorenstein ring. Then*

$$\Omega\mathrm{CM}(R) \text{ is of finite type} \iff \mathrm{CM}(E) \text{ is of finite type}$$

*where* $E = \mathrm{End}_R(\mathfrak{m}) \cong \mathfrak{m} : \mathfrak{m}$.

## Question 1.7

How about the case where $R$ is not an almost Gorenstein ring?

# 2. Main theorem

Note that $\mathfrak{m}$ is a regular reflexive trace ideal, once $R$ is not a DVR.

For an $R$-module $M$, consider the homomorphism

$$\tau : M^* \otimes_R M \to R,\ f \otimes m \mapsto f(m) \ \text{ for } f \in M^* \text{ and } m \in M$$

and set $\mathrm{tr}_R(M) = \mathrm{Im}\,\tau$.

We say that $I$ is a trace ideal of $R$ $\overset{def}{\iff}$ $I = \mathrm{tr}_R(M)$ for some $R$-module $M$
$\iff I = \mathrm{tr}_R(I)$
$\iff R : I = I : I.$ (when $I$ is regular)

- $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, if $R$ is not a DVR. (Goto-Matsuoka-Phoung)
- $M$ doesn't have free summands $\iff \mathrm{tr}_R(M) \subseteq \mathfrak{m}$. (Lindo)
- $I = R : A$ is a regular reflexive trace ideal of $R$.

Hence $\Omega\mathrm{CM}_{\mathcal{P}}(R) = \{M \in \Omega\mathrm{CM}(R) \mid \mathrm{tr}_R(M) \subseteq \mathfrak{m}\}$.

An $\mathfrak{m}$-primary ideal $I$ of $R$ is called

- Ulrich ideal if $I^2 = aI$ and $I/(a)$ is $R/I$-free for $\exists\, a \in I$
- good ideal if $I^2 = aI$ and $(a) :_R I = I$ for $\exists\, a \in I$.

For regular ideals in $R$, we have

$$\begin{array}{ccccccc} \text{Ulrich ideals} & \Longrightarrow & \text{good ideals} & \Longleftrightarrow & \text{stable trace ideals} & \Longrightarrow & \text{trace ideals} \\ & & \Big\Downarrow & & & & \\ & & \text{reflexive ideals} & & & & \end{array}$$

If $R$ is Gorenstein, there are one-to-one correspondences for regular ideals: (Goto-Isobe-Kumashiro, Goto-Isobe-T)

- $\{$ trace ideals $\}$ $\longleftrightarrow$ $\{$ birational module-finite extensions $\}$
- $\{$ good ideals $\}$ $\longleftrightarrow$ $\{$ Gorenstein birational module-finite extensions $\}$
- $\{$ Ulrich ideals $\}$ $\longleftrightarrow$ $\{$ Gorenstein birational extensions $A$ s.t. $\mu_R(A) = 2$ $\}$
- $\{$ reflexive trace ideals $\}$ $\longleftrightarrow$ $\{$ reflexive birational module-finite extensions $\}$

Let $I$ be a regular reflexive trace ideal of $R$. We set

- $A = \mathrm{End}_R(I) \cong I : I$
- $\Omega\mathrm{CM}(R, I) = \{M \in \Omega\mathrm{CM}(R) \mid \mathrm{tr}_R(M) \subseteq I\}$.

Choose $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$. Set $S = R[K]$ and $\mathfrak{c} = R : S$.

**Theorem 2.1 (Main theorem)**

(1) $\Omega\mathrm{CM}(A) \subseteq \Omega\mathrm{CM}(R, I) \subseteq \mathrm{CM}(A)$.

(2) $\Omega\mathrm{CM}(A) = \Omega\mathrm{CM}(R, I) \iff I$ *is stable.*

(3) $\Omega\mathrm{CM}(R, I) = \mathrm{CM}(A) \iff IK = I \iff I \subseteq \mathfrak{c}$.

**Corollary 2.2**

$\Omega\mathrm{CM}(A) = \mathrm{CM}(A) \iff$ *$I$ is stable and $I \subseteq \mathfrak{c}$* $\iff$ *$A$ is a Gorenstein ring.*

In particular, since $\Omega\mathrm{CM}(R, \mathfrak{c}) = \mathrm{CM}(S)$, we have

$$\Omega\mathrm{CM}(S) = \Omega\mathrm{CM}(R, \mathfrak{c}) \iff S \text{ is a Gorenstein ring.}$$

Recall $A = \mathrm{End}_R(I) \cong I : I = R : I \in \operatorname{mod} R$.

Note that

$$I \in \Omega\mathrm{CM}(R, I), \quad R \subseteq A \subseteq \overline{R}, \quad R : A = I, \quad \text{and} \quad R : (R : A) = A.$$

For $X, Y \in \operatorname{Mod} A$, we have

- $X$ is $A$-torsionfree $\iff$ $X$ is $R$-torsionfree
- $\mathrm{Hom}_R(X, Y) = \mathrm{Hom}_A(X, Y)$, provided $Y$ is $R$-torsionfree
- $\mathrm{tr}_R(X)A = \mathrm{tr}_R(X)$ and $\mathrm{tr}_R(X) \subseteq R : A$.

Let $M \in \Omega\mathrm{CM}(R)$ and $M \neq (0)$. Then

$$M \in \operatorname{Mod} A \text{ by extending the } R\text{-action} \iff \mathrm{tr}_R(M)A = \mathrm{tr}_R(M) \iff \mathrm{tr}_R(M) \subseteq R : A.$$

### Fact 2.3 (Kobayashi)

*Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in* $\operatorname{Mod} R$. *If* $M \in \Omega\mathrm{CM}_{\mathcal{P}}(R)$ *and* $N \in \mathrm{CM}(R)$, *then* $L \in \Omega\mathrm{CM}_{\mathcal{P}}(R)$.

(1) $\Omega\mathrm{CM}(A) \subseteq \Omega\mathrm{CM}(R, I) = \{M \in \Omega\mathrm{CM}(R) \mid \mathrm{tr}_R(M) \subseteq I\} \subseteq \mathrm{CM}(A)$.

(Proof) Let $M \in \Omega\mathrm{CM}(A)$. May assume $M \neq (0)$ and choose

$$0 \to M \to A^{\oplus n} \to A^{\oplus \ell} \quad \text{with} \quad n, \ell > 0.$$

- Suppose $R \lessdot_{\oplus} A^{\oplus n}$, so $A \cong R \oplus Y$ for $\exists\, Y \in \mathrm{Mod}\, R$.

Since $A/R$ is torsion, so is $Y$. This shows $Y = (0)$, because $Y \subseteq R \oplus Y \cong A$. Thus $A \cong R$. In particular, $\mu_R(A) = 1$. As $1 \notin \mathfrak{m}A$, we obtain $R = A$ and $I = R$. Therefore $M \in \Omega\mathrm{CM}(R)$ and $\mathrm{tr}_R(M) \subseteq R = I$.

- Suppose $R \not\lessdot_{\oplus} A^{\oplus n}$.

As $A^{\oplus n}$ is $R$-reflexive, we get $A^{\oplus n} \in \Omega\mathrm{CM}_{\mathcal{P}}(R)$. Consider

$$0 \to M \to A^{\oplus n} \to X \to 0 \quad \text{and} \quad 0 \to X \to A^{\oplus \ell}.$$

Then $X \in \mathrm{CM}(R)$, so $M \in \Omega\mathrm{CM}_{\mathcal{P}}(R) \subseteq \Omega\mathrm{CM}(R)$. Thus $\mathrm{tr}_R(M) \subseteq R : A = I$.

In each case, we have $\Omega\mathrm{CM}(A) \subseteq \Omega\mathrm{CM}(R, I)$.

Let $M \in \Omega\mathrm{CM}(R, I)$. May assume $M \neq (0)$. Since $\mathrm{tr}_R(M) \subseteq I = R : A$, we have $M \in \mathrm{Mod}\, A$. As $M$ is $R$-torsionfree, $M \in \mathrm{CM}(A)$. Hence $\Omega\mathrm{CM}(R, I) \subseteq \mathrm{CM}(A)$.

(3) $\Omega\mathrm{CM}(R, I) = \mathrm{CM}(A) \implies I \subseteq \mathfrak{c} = R : S$.

Note that $A = R : I = (K : K) : I = K : IK$. By setting $\mathrm{K}_A = K : A$, we have

$$\mathrm{K}_A = K : (K : IK) = IK.$$

Suppose $\Omega\mathrm{CM}(R, I) = \mathrm{CM}(A)$. Since $IK = \mathrm{K}_A \in \Omega\mathrm{CM}(R)$, we can choose

$$0 \to IK \xrightarrow{\varphi} R^{\oplus n} \to R^{\oplus \ell} \quad \text{where} \quad n > 0, \ \ell \geq 0.$$

Because $\mathrm{tr}_R(IK) \subseteq I$, we have $\mathrm{Im}\,\varphi \subseteq I^{\oplus n}$. This induces

$$\varphi' : IK \to I^{\oplus n}, \quad x \mapsto \varphi(x) \ \ (x \in IK).$$

Since $I^{\oplus n}$ is $A$-torsionfree, $\varphi' \in \mathrm{Hom}_R(IK, I^{\oplus n}) = \mathrm{Hom}_A(IK, I^{\oplus n})$. Look at

$$\begin{array}{ccccccccc}
0 & \longrightarrow & IK & \xrightarrow{\varphi} & R^{\oplus n} & \longrightarrow & X = \mathrm{Coker}\varphi & \longrightarrow & 0 \quad \text{in} \ \ \mathrm{Mod}\,R \\
 & & \| & & \uparrow i & & \uparrow \exists 1\, h & & \\
0 & \longrightarrow & IK & \xrightarrow{\varphi'} & I^{\oplus n} & \longrightarrow & X' = \mathrm{Coker}\varphi' & \longrightarrow & 0 \quad \text{in} \ \ \mathrm{Mod}\,A.
\end{array}$$

Then $\operatorname{depth}_R X' > 0$, so we have

$$0 \to \operatorname{Hom}_A(X', IK) \to \operatorname{Hom}_A(I^{\oplus n}, IK) \to \operatorname{Hom}_A(IK, IK) \to 0 = \operatorname{Ext}_A^1(X', IK).$$

This shows $\varphi' : IK \to I^{\oplus n}$ is split. By choosing $X \in \operatorname{Mod} A$ s.t. $I^{\oplus n} \cong IK \oplus X$,

$$\begin{aligned} \mathrm{M}_n(A) &= \mathrm{M}_n(I : I) \cong \mathrm{M}_n(\operatorname{End}_A(I, I)) \cong \operatorname{End}_A(I^{\oplus n}) \cong \operatorname{End}_A(IK \oplus X) \\ &\cong \begin{bmatrix} \operatorname{Hom}_A(IK, IK) & \operatorname{Hom}_A(X, IK) \\ \operatorname{Hom}_A(IK, X) & \operatorname{Hom}_A(X, X) \end{bmatrix}. \end{aligned}$$

Then $Y = \operatorname{Hom}_A(X, IK)$ is $A$-projective. For each $\mathfrak{p} \in \operatorname{Ass} A$, we have

$$A_\mathfrak{p} \oplus X_\mathfrak{p} \cong (K_A)_\mathfrak{p} \oplus X_\mathfrak{p} \cong (IK \oplus X)_\mathfrak{p} \cong (I^{\oplus n})_\mathfrak{p} \cong (IA_\mathfrak{p})^{\oplus n} = A_\mathfrak{p}^{\oplus n}.$$

Hence $X_\mathfrak{p} \in \operatorname{mod} A$ is a free $A_\mathfrak{p}$-module of rank $n - 1$. This induces

$$Y_\mathfrak{p} \cong \operatorname{Hom}_{A_\mathfrak{p}}(X_\mathfrak{p}, (IK)_\mathfrak{p}) \cong \operatorname{Hom}_{A_\mathfrak{p}}(X_\mathfrak{p}, A_\mathfrak{p}) \cong A_\mathfrak{p}^{\oplus(n-1)}$$

so that $Y \cong A^{\oplus(n-1)}$ in $\operatorname{Mod} A$.

Therefore

$$X \cong \mathrm{Hom}_A(\mathrm{Hom}_A(X, \mathrm{K}_A), \mathrm{K}_A) \cong \mathrm{Hom}_A(A^{\oplus(n-1)}, \mathrm{K}_A) \cong \mathrm{K}_A^{\oplus(n-1)}$$

whence $I^{\oplus n} \cong \mathrm{K}_A \oplus X \cong \mathrm{K}_A^{\oplus n}$. Since $IA_P$ is a faithful ideal with $\mathrm{id}_{A_P} IA_P = 1$, we see that $IA_P \cong (\mathrm{K}_A)_P$ for $\forall\, P \in \mathrm{Max}\, A$. Hence

$$\widehat{A} \otimes_A I \cong \widehat{A} \otimes_A \mathrm{K}_A \quad \text{in} \ \ \mathrm{Mod}\, \widehat{A}.$$

Therefore $I \cong \mathrm{K}_A = IK$ in $\mathrm{Mod}\, A$, so that

$$IK = \alpha I \quad \text{for} \ \ \exists\, \alpha \in \mathrm{Q}(R)^{\times}.$$

Thus, for all $\ell > 0$, the equality $IK^{\ell} = \alpha^{\ell} I$ holds. Recall $S = R[K]$. By choosing $\ell \gg 0$ with $S = K^{\ell}$, we have $IS = \alpha^{\ell} I$, so that

$$\alpha^{\ell} I = IS = IS \cdot S = \alpha^{\ell} IS.$$

This implies $I = IS$, whence $I \subseteq I : S \subseteq R : S = \mathfrak{c}$. □

For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, we denote by

- $\operatorname{ind}\mathcal{X}$ the set of isomorphism classes of indecomposable $R$-modules in $\mathcal{X}$.

## Corollary 2.4

*Let $R$ be a Gorenstein local domain with* $\dim R = 1$. *Then*

$$\begin{aligned} \operatorname{ind}\Omega\mathrm{CM}(R) &= \bigcup_{R \neq A \in \mathcal{Y}} \operatorname{ind}\mathrm{CM}(A) \cup \{[R]\} \\ &= \bigcup_{I \in \mathcal{T},\, I \neq R} \operatorname{ind}\mathrm{CM}(\mathrm{End}_R(I)) \cup \{[R]\} \end{aligned}$$

where

- $\mathcal{Y}$ is the set of birational module-finite extensions $A$ s.t. $A \in \mathrm{Ref}(R)$
- $\mathcal{T}$ is the set of regular reflexive trace ideals of $R$.

## Question 2.5

$\Omega\mathrm{CM}(R)$ is of finite type $\iff$ $\mathrm{CM}(A)$ is of finite type for some $A \in \mathcal{Y}$?

# 3. When is $\Omega\mathrm{CM}(R)$ of finite type?

Recall $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$, $S = R[K]$ and $\mathfrak{c} = R : S$. Then $S \in \mathcal{Y}$ and

- $R$ is a Gorenstein ring $\iff R = K \iff R = S \iff R = \mathfrak{c}$
- $R$ is an almost Gorenstein ring $\iff K/R \cong (R/\mathfrak{m})^{\oplus} \iff S/R \cong (R/\mathfrak{m})^{\oplus}$ $\iff \mathfrak{m} \subseteq \mathfrak{c}$
- $R$ is an generalized Gorenstein ring if $R = \mathfrak{c}$, or $R \neq \mathfrak{c}$ and $K/R$ is $R/\mathfrak{c}$-free.

## Theorem 3.1

*Suppose $R$ is a generalized Gorenstein ring with minimal multiplicity. Then*

$$|\operatorname{ind}\Omega\mathrm{CM}(R)| = \ell_R(R/\mathfrak{c}) + |\operatorname{ind}\mathrm{CM}(S)|.$$

*Hence, $\Omega\mathrm{CM}(R)$ is of finite type $\iff$ $\mathrm{CM}(S)$ is of finite type.*

## Corollary 3.2

*Suppose $\mathrm{e}(R) = v(R) = 3$. Then $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \ell_R(R/\mathfrak{c}) + |\operatorname{ind}\mathrm{CM}(S)|$.*

## Corollary 3.3 (cf. Kobayashi)

*Suppose $R$ is a non-Gorenstein almost Gorenstein ring with minimal multiplicity. Then* $|\operatorname{ind}\Omega\mathrm{CM}(R)| = 1 + |\operatorname{ind}\mathrm{CM}(E)|$, *where* $E = \mathfrak{m} : \mathfrak{m}$.

## Proposition 3.4

*Suppose $\overline{R}$ is a DVR, $\overline{R} \in \operatorname{mod} R$, and $\mathfrak{m}\overline{R} \subseteq R$. Then* $\operatorname{ind}\Omega\mathrm{CM}(R) = \{[R], [\overline{R}]\}$.

## Example 3.5

Let $A$ be a RLR with $n = \dim A \geq 2$. Let $X_1, X_2, \ldots, X_n$ be a regular sop of $A$. We set

$$R = A/\bigcap_{i=1}^{n}(X_j \mid 1 \leq j \leq n,\ j \neq i).$$

Then $\operatorname{ind}\Omega\mathrm{CM}(R) = \{[R], [\overline{R}]\}$.

## Example 3.6

Suppose $\operatorname{ch} R > 0$ and $\overline{R}$ is a DVR. If $R$ is $F$-pure, then $\operatorname{ind}\Omega\mathrm{CM}(R) = \{[R], [\overline{R}]\}$.

Note that

- if $R$ is a generalized Gorenstein ring with minimal multiplicity, then $S = R[K]$ is a Gorenstein ring.

## Corollary 3.7

*Let $R$ be the numerical semigroup ring over a field $k$. Suppose that $R$ is a generalized Gorenstein ring with minimal multiplicity. Then TFAE.*

(1) $\Omega\mathrm{CM}(R)$ *is of finite type.*

(2) $S = k[[H]]$ *is a semigroup ring of $H$, where $H$ is one of the following forms:*

(a) $H = \mathbb{N}$,
(b) $H = \langle 2, 2q+1 \rangle$ $(q \geq 1)$,
(c) $H = \langle 3, 4 \rangle$, *or*
(d) $H = \langle 3, 5 \rangle$.

## Example 3.8 (Kobayashi)

Let $R = k[[t^3, t^7, t^8]]$. Then $E = \mathfrak{m} : \mathfrak{m} = k[[t^3, t^4, t^5]]$, so $\mathrm{CM}(E)$ is of finite type. Thus $\Omega\mathrm{CM}(R)$ is of finite type, but $R$ is not an almost Gorenstein ring.

Since $\mathrm{e}(R) = 3$, $R$ is a generalized Gorenstein ring.
Note that $K = R + Rt \cong \mathrm{K}_R$ and $R \subseteq K \subseteq \overline{R}$. Hence

$$S = R[K] = R[t] = k[[t]] = k[[\mathbb{N}]] = \overline{R} =: V$$

which yields $\Omega\mathrm{CM}(R)$ is of finite type. Moreover

- $\mathfrak{c} = R : S = R : V = t^6 V$
- $\ell_R(R/\mathfrak{c}) = \ell_R(V/\mathfrak{c}) - \ell_R(V/R) = 6 - 4 = 2.$

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |
| 9 | 10 | 11 |
| 12 | 13 | 14 |

Therefore

$$|\operatorname{ind}\Omega\mathrm{CM}(R)| = 2 + |\operatorname{ind}\mathrm{CM}(S)| = 3.$$

Note that if $\mathrm{CM}(R)$ is of finite type, then

- $\mathcal{X}_R$ is a finite set (Goto-Ozeki-Takahashi-Watanabe-Yoshida)
- $R$ is analytically unramified (Krull, Leuschke-Wiegand)

where $\mathcal{X}_R$ denotes the set of Ulrich ideals of $R$.

### Theorem 3.9

*If $\Omega\mathrm{CM}(R)$ is of finite type, then $\mathcal{X}_R$ is finite and $R$ is analytically unramified.*

### Example 3.10

Let $(A, \mathfrak{m})$ be a CM local ring with $\dim A = 1$, $\exists\, \mathrm{K}_A$, $|A/\mathfrak{m}| = \infty$. Assume $\mathrm{Q}(A)$ is a Gorenstein ring. We set

$$R = A \ltimes A.$$

Then, because $|\mathcal{X}_R| = \infty$, we have $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \infty$.

We say that $R$ is an Arf ring, if every integrally closed regular ideal is stable.

### Theorem 3.11 (cf. Dao, Dao-Lindo, Isobe-Kumashiro)

*Suppose $\overline{R}$ is a local ring. If $R$ is an analytically unramified Arf ring, then $\Omega\mathrm{CM}(R)$ is of finite type. In particular, $\mathcal{X}_R$ is finite.*

### Example 3.12

Let $R = k[[t^3, t^4]]$. Then $|\operatorname{ind}\Omega\mathrm{CM}(R)| = |\operatorname{ind}\mathrm{CM}(R)| < \infty$, but $R$ is not an Arf ring.

### Example 3.13

Let $R = k[[t^3, t^7]]$. Then

$$|\mathcal{X}_R| = |\{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}| < \infty$$

provided $k$ is finite. However $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \infty$ and $R$ is not an Arf ring.

**Thank you for your attention.**